Blow-up analysis for some mean field equations involving probability measures from statistical hydrodynamics

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Abstract

Motivated by the mean field equations with probability measure derived by Sawada-Suzuki and by Neri in the context of the statistical mechanics description of two-dimensional turbulence, we study the semilinear elliptic equation with probability measure:

$$-\Delta v = \lambda \int_I V(\alpha, x, v) e^{\alpha v}\, \mathcal{P}(d\alpha) - \frac{\lambda}{|\Omega|} \iint_{I\times\Omega} V(\alpha, x, v) e^{\alpha v}\, \mathcal{P}(d\alpha) dx,$$

defined on a compact Riemannian surface. This equation includes the above mentioned equations of physical interest as special cases. For such an equation we study the blow-up properties of solution sequences. The optimal Trudinger-Moser inequality is also considered.

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1 Introduction

Motivated by several mean field equations recently derived in the context of Onsager's statistical mechanics description of turbulence [14], we study concentrating sequences of solutions to the following equation:

$$-\Delta v = \lambda \int_I V(\alpha, x, v) e^{\alpha v}\, \mathcal{P}(d\alpha) - \frac{\lambda}{|\Omega|} \iint_{I\times\Omega} V(\alpha, x, v) e^{\alpha v}\, \mathcal{P}(d\alpha) dx, \tag{1.1}$$

where Ω is a compact two-dimensional orientable Riemannian manifold without boundary, $I = [-1, 1]$, $\mathcal{P} \in \mathcal{M}(I)$ is a Borel measure, $v \in H^1(\Omega)$ is a function normalized by $\int_\Omega v = 0$, $\lambda > 0$ and $V(\alpha, x, v)$ is a *functional* satisfying the condition $\alpha V(\alpha, x, v) \geq 0$, as well as suitable bounds which will be specified below.

A typical special case of physical interest is given by

$$V(\alpha, x, v) = V_1(\alpha, x, v) = \frac{\alpha}{\int_\Omega e^{\alpha v}\, dx},$$

in which case equation (1.1) reduces to the mean field equation derived by Sawada and Suzuki in [17]:

$$-\Delta v = \lambda \int_I \alpha \left(\frac{e^{\alpha v}}{\int_\Omega e^{\alpha v}\, dx} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\alpha). \tag{1.2}$$

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In order to relate our results to the literature, we note that under the further assumption $\mathcal{P} = \delta_1$, the Dirac concentrated at $\alpha = 1$, equation (1.2) reduces to the well known mean field equation

$$-\Delta v = \lambda \left(\frac{e^v}{\int_\Omega e^v \, dx} - \frac{1}{|\Omega|} \right) \tag{1.3}$$

extensively studied in recent years. See, e.g., [20] and the references therein for results and applications of (1.3) to physics, biology and geometry. Assuming instead that

$$\mathcal{P} = t\delta_1 + (1-t)\delta_{-1}, \tag{1.4}$$

equation (1.2) reduces to the mean field sinh-Gordon type equation derived in [6, 15]. Several blow-up results for (1.2)–(1.4) have been obtained in recent years by Ohtsuka and Suzuki in [11, 12], and applied to derive the best constant for the corresponding Trudinger-Moser inequality. A construction of two-sided blow up solutions was obtained in [2]. A blow-up analysis for (1.2) is contained in [9], and the best constant for the corresponding Trudinger-Moser inequality will appear in [16].

Another special case of physical interest, which is the main motivation to this work, is given by

$$V(\alpha, x, v) = V_2(\alpha, x, v) = \frac{\alpha}{\iint_{I\times\Omega} e^{\alpha v} \, \mathcal{P}(d\alpha)}. \tag{1.5}$$

In this case, equation (1.1) reduces to the mean field equation derived by Neri [8]:

$$-\Delta v = \lambda \frac{\int_I \alpha (e^{\alpha v} - \frac{1}{|\Omega|}\int_\Omega e^{\alpha v} dx) \, \mathcal{P}(d\alpha)}{\iint_{I\times\Omega} e^{\alpha v} \, \mathcal{P}(d\alpha) dx}. \tag{1.6}$$

An existence result for solutions to equation (1.6) under Dirichlet boundary conditions was also obtained in [8]. In view of the results in [9], it is natural to study the concentrating sequences of solutions to (1.6). Actually, since both equations are motivated by the same physical problem, it is natural to compare these equations and to seek their common features as well as their differences. Taking this point of view, we study the general mean field equation (1.1) and we derive blow-up properties which are common to equation (1.2) and equation (1.6). On the other hand, we will show that from the point of view of the Trudinger-Moser inequality, the two equations exhibit different properties. Indeed, while equation (1.2) leads to an improved best constant with respect to (1.3), somewhat unexpectedly such a situation does *not* occur for equation (1.6).

We organize this article as follows. In Section 2 we state our main blow-up results for (1.1), namely Theorem 2.1 and Theorem 2.2. We note that some results, such as (2.3), are new even for equation (1.2). In Section 3 we carry out the blow-up analysis. Although we follow the approach in [9], based on the consideration of measures defined on the product space $I \times \Omega$, some technical lemmas are stated under weaker and more natural assumptions. In Section 4 we apply our results to the special cases of physical interest. We prove some results specific to Neri's equation (1.6), particularly in relation to the residual vanishing property and the optimal Trudinger-Moser inequality, see Theorem 4.1 and Theorem 4.2, respectively.

Notation. In what follows, we denote by C a general constant whose value may change from line to line. For all $p \in \Omega$ we denote by $\delta_p \in \mathcal{M}(\Omega)$ the Dirac measure centered at p. For all $\alpha \in I$ we denote by $\delta_\alpha \in \mathcal{M}(I)$ the Dirac measure centered at α. We denote by dx the volume element on Ω and by $|\Omega|$ the volume of Ω. When the integration variable is clear from the context, for simplicity we omit it.

2 Main results

We define

$$\mathcal{E} = \left\{ v \in H^1(\Omega) : \int_\Omega v = 0 \right\}.$$

and we make the following assumptions on the functional V.

(V1) $(\mathrm{sign}\,\alpha)\, V(\alpha, x, v) \geq 0$ for all $(\alpha, x, v) \in I \times \Omega \times \mathcal{E}$;

(V2) $\sup_{\mathcal{E}} \|V(\alpha, x, v(x))\|_{L^\infty(I\times\Omega)} \leq C_1$ for some constant $C_1 > 0$;

(V3) $\iint_{I\times\Omega} |V(\alpha, x, v)| e^{\alpha v}\, \mathcal{P}(d\alpha) dx \leq C_2$ for some constant $C_2 > 0$.

We consider solution sequences $\{v_n\}$, $\lambda_n \to \lambda_0$ to

$$\begin{cases} -\Delta v_n = \lambda_n \displaystyle\int_I \left(V(\alpha, x, v_n) e^{\alpha v_n} - \frac{1}{|\Omega|} \int_\Omega V(\alpha, x, v_n) e^{\alpha v_n}\, dx \right) \mathcal{P}(d\alpha) \\ \displaystyle\int_\Omega v_n = 0. \end{cases} \tag{2.1}$$

Following the approach of Brezis and Merle [1], see also Nagasaki and Suzuki [7], we begin by proving that the blow-up set for concentrating solutions is finite and that a "minimum mass" is necessary for blow-up to occur. We define the blow-up sets:

$$\mathcal{S}_\pm = \{p \in \Omega : \exists p_{\pm,n} \to p : v_n(p_{\pm,n}) \to \pm\infty)\}$$

and denote $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$. We define the measures $\nu_{\pm,n} \in \mathcal{M}(\Omega)$ by setting

$$\nu_{\pm,n} = \lambda_n \int_{I_\pm} |V(\alpha, x, v_n)| e^{\alpha v_n}\, \mathcal{P}(d\alpha) \tag{2.2}$$

where $I_+ = [0,1]$ and $I_- = [-1,0)$. Since in view of (V3) we have $\int_\Omega \nu_{\pm,n} \leqslant C_2 \lambda_n$, we may assume that $\nu_{\pm,n} \overset{*}{\rightharpoonup} \nu_\pm$ for some measure $\nu_\pm \in \mathcal{M}(\Omega)$.

Theorem 2.1. *Assume (V1)–(V2)–(V3). Let v_n be a solution sequence to (2.1) with $\lambda_n \to \lambda_0$. Then, the following alternative holds.*

i) Compactness: $\limsup_{n\to\infty} \|v_n\|_\infty < +\infty$. *There exist a solution $v \in \mathcal{E}$ to (1.1) with $\lambda = \lambda_0$ and a subsequence $\{v_{n_k}\}$ such that $v_{n_k} \to v$ in $\mathcal{E}$.*

ii) Concentration: $\limsup_{n\to\infty} \|v_n\|_{L^\infty} = +\infty$. *The sets $\mathcal{S}_\pm$ are finite and $\mathcal{S} = \mathcal{S}_- \cup \mathcal{S}_+ \neq \emptyset$. For some $s_\pm \geqslant 0$, $s_\pm \in L^1(\Omega)$ we have*

$$\nu_\pm = s_\pm dx + \sum_{p\in\mathcal{S}_\pm} n_{\pm,p} \delta_p$$

with $n_{\pm,p} \geqslant 4\pi$ for all $p \in \mathcal{S}$. Moreover, there exist $v \in H^1_{\mathrm{loc}}(\Omega \setminus \mathcal{S})$, $k \in L^\infty(I \times \Omega)$ and $c_0 \in \mathbb{R}$ such that $v_n \to v$ in $H^1_{\mathrm{loc}}(\Omega \setminus \mathcal{S})$ and

$$\begin{cases} -\Delta v = \lambda_0 \displaystyle\int_I k(\alpha, x) e^{\alpha v} \mathcal{P}(d\alpha) + \sum_{p\in\mathcal{S}_+} n_{+,p}\delta_p - \sum_{p\in\mathcal{S}_-} n_{-,p}\delta_p - c_0 \qquad \text{in } \Omega, \\ \displaystyle\int_\Omega v = 0. \end{cases} \tag{2.3}$$

Under stronger assumptions on V, the blow-up results may be refined. Following [9] we consider measures defined on the product space $I \times \Omega$. We assume that V does not depend on x, namely $V = V(\alpha, v)$ and

(V0) $\nabla_x V(\alpha, v) = 0$.

We also strengthen assumptions (V2)–(V3) above as follows:

(V2') $\sup_{\mathcal{E}} \|\alpha^{-1} V(\alpha, v)\|_{L^\infty(I)} \le C_1'$ for some constant $C_1' > 0$;

(V3') $\iint_{I\times\Omega} |\alpha^{-1} V(\alpha, v)| e^{\alpha v}\, \mathcal{P}(d\alpha) dx \le C_2'$ for some constant $C_2' > 0$.

For every fixed $\alpha \in I$ we define $\mu_\alpha^n(dx) \in \mathcal{M}(\Omega)$ by setting

$$\mu_\alpha^n(dx) = \lambda_n \frac{V(\alpha, v_n)}{\alpha} e^{\alpha v_n}\, dx. \tag{2.4}$$

We consider the sequence of measures $\mu_n = \mu_n(d\alpha dx) \in \mathcal{M}(I \times \Omega)$ defined by

$$\mu_n(d\alpha dx) = \mu_\alpha^n(dx)\mathcal{P}(d\alpha) = \lambda_n \frac{V(\alpha, v_n)}{\alpha} e^{\alpha v_n}\, dx\mathcal{P}(d\alpha). \tag{2.5}$$

In view of (V3'), for large values of n we have:

$$\mu_n(I \times \Omega) = \iint_{I\times\Omega} \mu_\alpha^n(dx)\mathcal{P}(d\alpha) \leqslant C_2'(\lambda_0 + 1).$$

Hence, upon extracting a subsequence, we may assume that

$$\mu_n \stackrel{*}{\rightharpoonup} \mu \text{ for some Borel measure } \mu \in \mathcal{M}(I \times \Omega). \tag{2.6}$$

In the next result we describe some properties of μ.

Theorem 2.2. *Suppose that V satisfies (V0)–(V1)–(V2')–(V3'). Let v_n be a solution sequence to* (2.1) *with $\lambda_n \to \lambda_0$. The following properties hold.*

(i) The singular part of μ has a "separation of variables" form:

$$\mu(d\alpha dx) = \sum_{p\in\mathcal{S}} \zeta_p(d\alpha)\delta_p(dx) + r(\alpha, x)\mathcal{P}(d\alpha)dx. \tag{2.7}$$

Here, $\zeta_p \in \mathcal{M}(I)$ and $r \in L^1(I \times \Omega)$.

(ii) For every $p \in \mathcal{S}$ the following relation is satisfied

$$8\pi \int_I \zeta_p(d\alpha) = \left[\int_I \alpha\zeta_p(d\alpha)\right]^2. \tag{2.8}$$

(iii) For every $p \in \mathcal{S}$ it holds

$$\int_{I_\pm} |\alpha|\zeta_p(d\alpha) = n_{\pm,p} \qquad \int_{I_\pm} |\alpha| r(\alpha, x)\mathcal{P}(d\alpha) = s_\pm(x),$$

where $n_{\pm,p}$ and $s_\pm(x)$ are as in Theorem 2.1. Moreover, for every $p \in \mathcal{S}_\pm \setminus \mathcal{S}_\mp$

$$\int_{I_\mp} |\alpha|\zeta_p(d\alpha) = 0.$$

3 A blow-up analysis

We begin by recalling some preliminary results. We first provide an extension of a key result from [1] to the case of potentials defined on product spaces, following the approach in [9]. We actually weaken the assumptions in [9] and derive a somewhat more natural formulation. Let $D \subset \mathbb{R}^2$ be a bounded domain and for every $a \in \mathbb{R}$ let a^+ be the positive part of a, $a^+ = \max\{a, 0\}$.

Lemma 3.1. *Let (u_n) be a solution sequence to*

$$-\Delta u_n = \int_{I_+} W_n(\alpha, x) e^{\alpha u_n} \mathcal{P}(d\alpha) \qquad \text{in } D,$$

where $W_{\alpha,n} \geq 0$ verifies $\| \int_{I_+} W_{\alpha,n} \mathcal{P}(d\alpha) \|_{L^p(D)} \leqslant C$, $p \in (1, \infty]$ and $\|u_n^+\|_{L^1(D)} \leqslant C$. Suppose that for every $n \in \mathbb{N}$ we have

$$\iint_{D \times I_+} W_n(\alpha, x) e^{\alpha u_n} \mathcal{P}(d\alpha) dx \leqslant \varepsilon_0 < \frac{4\pi}{p'}, \tag{3.1}$$

where $p' = p/(p-1)$ is the conjugate exponent to p. Then, $\{u_n^+\}$ is bounded in $L^\infty_{loc}(D)$.

Proof. Without loss of generality we may assume that $D = B_R$. We split u_n as $u_{1n} + u_{2n}$ where u_{1n} is the solution of

$$\begin{cases} -\Delta u_{1n} = \displaystyle\int_{I_+} W_n(\alpha, x) e^{\alpha u_n} \text{ in } B_R \\ u_{1n} = 0 \text{ on } \partial B_R \end{cases} \tag{3.2}$$

so that $\Delta u_{2n} = 0$ in B_R. By the mean value theorem for harmonic functions, (3.2) and assumption (3.1) we have

$$\|u_{2n}^+\|_{L^\infty(B_{R/2})} \leqslant C \|u_{2n}^+\|_{L^1(B_R)} \leqslant C \left[\|u_n^+\|_{L^1(B_R)} + \|u_{1n}\|_{L^1(B_R)} \right] \leqslant C$$

We define

$$\varphi_n = \int_{I_+} W_n(\alpha, x) e^{\alpha u_n} \mathcal{P}(d\alpha)$$

so that, by (3.1) we have

$$\|\varphi_n\|_{L^1(B_R)} \leqslant \varepsilon_0 < \frac{4\pi}{p'}. \tag{3.3}$$

By [1], Theorem 1, for any $\delta \in (0, 4\pi)$ we have

$$\int_D \exp\left[\frac{(4\pi - \delta)|u_{1n}(x)|}{\|\varphi\|_{L^1}} \right] dx \leqslant \frac{4\pi^2}{\delta} (\text{diam } D)^2.$$

Moreover, since $\varepsilon_0 < \frac{4\pi}{p'}$ there exists $\delta_0 \in (0, 4\pi)$ such that $\varepsilon_0 = \frac{4\pi - \delta_0}{p'}$. Then, for $\bar{\delta} \in (0, \delta_0)$, using (3.3) we have

$$\int_{B_{R/2}} \exp\left[(p' + \eta)|u_{1n}(x)| \right] dx \leqslant \int_{B_{R/2}} \exp\left[\frac{(4\pi - \bar{\delta})|u_{1n}(x)|}{\|\varphi\|_{L^1(B_R)}} \right] dx \leqslant C \tag{3.4}$$

where $\eta = \frac{\delta_0 - \bar{\delta}}{4\pi - \delta_0} p'$. Hence, the sequence $\{e^{|u_{1n}|}\}$ is bounded in $L^{p'+\eta}(B_R)$ so that the sequence $\{e^{u_n^+}\}$ is bounded in $L^{p'+\eta}(B_{R/2})$ for some $\eta > 0$. On the other hand,

$$\begin{aligned}\int_{B_{R/2}} \left| \int_{I_+} W_n(\alpha, x) e^{\alpha u_n} \mathcal{P}(d\alpha) \right|^r dx &\leqslant \int_{B_{R/2}} e^{r u_n^+} \left(\int_{I_+} W_n(\alpha, x) \mathcal{P}(d\alpha) \right)^r dx \\ &\leqslant \left(\int_{B_{R/2}} e^{\frac{pr}{p-r} u_n^+} \right)^{\frac{p-r}{p}} \left(\int_{B_{R/2}} \left(\int_{I_+} W_n(\alpha, x) \mathcal{P}(d\alpha) \right)^p dx \right)^{\frac{r}{p}} \\ &= \| e^{u_n^+} \|^r_{L^{\frac{pr}{p-r}}(B_{R/2})} \left\| \int_{I_+} |W_{\alpha,n}| \mathcal{P}(d\alpha) \right\|^r_{L^p(B_{R/2})}.\end{aligned}$$

If we choose $r \in (1, p)$ in order to have $pr/(p-r) = p' + \eta$, by (3.2) and the elliptic estimates we see that u_{1n} is bounded in $L^\infty(B_{R/4})$. Therefore $\{u_n^+\}$ is bounded in $L^\infty(B_{R/4})$. $\square$

Now we recall the following result for equations defined on manifolds obtained in [9] (see also [10]). Let (Ω, g) be a Riemannian surface. We consider solution sequences $\{u_n\}$ to the equation

$$-\Delta u_n = \int_{I_+} W_n(\alpha, x) e^{\alpha u_n} \mathcal{P}(d\alpha) + f_n \qquad \text{on } \Omega \tag{3.5}$$

and set

$$\sigma_n = \int_{I_+} W_n(\alpha, x) e^{\alpha u_n} \mathcal{P}(d\alpha).$$

We weaken the assumptions in [9] by assuming uniform boundedness of

$$\|W_n(\alpha, x)\|_{L^p(D; L^1(I_+))}$$

with respect to n. We recall that $\|W_n(\alpha, x)\|_{L^\infty(D; L^1(I_+))} \leq C$ was assumed in [9].

Lemma 3.2. *Let $\{u_n\}$ be a solution sequence to* (3.5) *where*

$$\left\| \int_{I_+} W_n(\alpha, x) \mathcal{P}(d\alpha) \right\|_{L^p} \leqslant C,$$

$W_n(\alpha, x) \geq 0$, $\|f_n\|_\infty \leqslant C$ and $\|u_n^+\|_1 \leqslant C$. Suppose that $\sigma_n \overset{*}{\rightharpoonup} \sigma$ and $\sigma(\{x_0\}) < 4\pi/p'$ for some $x_0 \in \Omega$. Then, there exists a neighborhood $\widetilde{U} \subset \Omega$ of x_0 such that

$$\limsup_{n \to \infty} \|u_n^+\|_{L^\infty(\widetilde{U})} < +\infty.$$

Proof. Let (U, ψ) a local isothermal chart such that $\psi(x_0) = 0$, $g = e^{\xi(X)}(dX_1^2 + dX_2^2)$. Then, $u_n(X) = u_n(\psi^{-1}(X))$ satisfies

$$-\Delta_X u_n = \left(\int_{I_+} W(\alpha, n) e^{\alpha u_n} \mathcal{P}(d\alpha) + f_n \right) e^\xi \qquad \text{in } D = \psi(U).$$

Let h_n be defined by

$$-\Delta h_n = f_n e^\xi \text{ in } D, \qquad\qquad h_n = 0 \text{ on } \partial D.$$

Then, $\|h_n\|_{L^\infty(D)} \leq C$ and $\widetilde{u}_n = u_n - h_n$ satisfies

$$-\Delta \widetilde{u}_n = e^\xi \int_{I_+} W(\alpha, x) e^{h_n} e^{\alpha \widetilde{u}_n} \mathcal{P}(d\alpha) \qquad \text{in } D.$$

On the other hand, setting $\widetilde{W}_n(\alpha, x) = e^{\xi} W(\alpha, x) e^{h_n}$ we have

$$\|\widetilde{W}_n(\alpha, x)\|_{L^p(D;L^1(I_+))} \le \|e^{\xi} e^{h_n}\|_\infty \|W_n(\alpha, x)\|_{L^p(D;L^1(I_+))} \le C.$$

Moreover,

$$\|\widetilde{u}_n^+\|_{L^1(D)} \le \|u_n^+\|_{L^1(\Omega)} + |D| \|h_n\|_{L^\infty(D)} \le C$$

and

$$\int_D e^{\xi} \int_{I_+ W(\alpha,x) e^{\alpha h_n}} e^{\alpha \widetilde{u}_n} \, \mathcal{P}(d\alpha) dX = \sigma_n(U).$$

In view of the assumption, there exists $U' \subset U$, $x_0 \in U'$ such that

$$\iint_{I_+ \times U'} W(\alpha, x) e^{\alpha u_n} \, \mathcal{P}(d\alpha) dx \le \varepsilon_0 < \frac{4\pi}{p'}.$$

In view of Lemma 3.1, $\widetilde{u}_n$ is bounded in $L^\infty_{\mathrm{loc}}(\psi^{-1}(U'))$. Taking $\widetilde{U} \Subset U'$ we conclude the proof. $\square$

We can now prove our first result.

Proof of Theorem 2.1. We denote by $G = G(x, y)$ the Green's function associated to $-\Delta$ on Ω. Namely, G is defined by

$$\begin{cases} -\Delta_x G(x, y) = \delta_y - \frac{1}{|\Omega|} \\ \int_\Omega G(x, y) dx = 0. \end{cases} \tag{3.6}$$

For every solution v_n to (2.1) we define

$$u_{\pm,n}(x) = G \star \nu_{\pm,n}(x) = \int_\Omega G(x, y) \nu_{\pm,n}(y) dy$$

where $\nu_{\pm,n}$ is defined in (2.2). Then, $v_n = u_{+,n} - u_{-,n}$. We observe that $u_{\pm,n}$ is uniformly bounded below. Indeed, let $A > 0$ be such that $G(x, y) \ge -A$ for all $x, y \in \Omega$. Then,

$$u_{\pm,n}(x) = \int_\Omega G(x, y) \nu_{\pm,n}(y) \, dy \ge -A \int_\Omega \nu_{\pm,n}(y) \, dy \ge -AC_2 \lambda_n \ge -AC_2(\lambda_0 + 1).$$

In this sense, we say that $u_{+,n}$ is the "positive part" of v_n and $u_{-,n}$ is the "negative part" of v_n. Furthermore, in view of Assumption (V1), the functions $u_{\pm,n}$ satisfy the Liouville system:

$$\begin{cases} -\Delta u_{\pm,n} = \lambda_n \displaystyle\int_{I_\pm} |V(\alpha, x, v_n)| e^{|\alpha|(u_{\pm,n} - u_{\mp,n})} \, \mathcal{P}(d\alpha) - c_{\pm,n} \\ \displaystyle\int_\Omega u_{\pm,n} \, dx = 0 \end{cases} \tag{3.7}$$

where

$$c_{\pm,n} = \frac{\lambda_n}{|\Omega|} \iint_{I_\pm \times \Omega} |V(\alpha, x, v_n)| e^{\alpha v_n} \, \mathcal{P}(d\alpha).$$

We check that the equations in (3.7) satisfy the assumptions of Lemma 3.2 with $W_n(\alpha, x) = |V(\alpha, v_n(x))| e^{-\alpha u_{-,n}(x)}$, $p = \infty$ and $f_n = c_{+,n}$. To this end, we note that in view of Assumption (V2), we have, for every $\alpha \in I_+$:

$$V(\alpha, x, v_n) e^{-\alpha u_{-,n}} \le V(\alpha, x, v_n) e^{AC_2(\lambda_0+1)} \le C_1 e^{AC_2(\lambda_0+1)}.$$

Therefore, setting $W_n(\alpha, x) = |V(\alpha, v_n(x))|e^{-\alpha u_{-,n}(x)}$ we have

$$0 \le W_n(\alpha, x) \le C_1 e^{AC_2(\lambda_0+1)}.$$

Furthermore,

$$c_{+,n} \le \frac{\lambda_n}{|\Omega|} \iint_{I_+\times\Omega} |V(\alpha, x, v_n)| e^{u_{+,n}-u_{-,n}}\, \mathcal{P}(d\alpha) \le C_2 \frac{\lambda_0 + 1}{|\Omega|}.$$

Let

$$\mathcal{S}_{u_+} = \{p \in \Omega : \nu_+(\{p\}) \geqslant 4\pi\}.$$

Since $\nu_{+,n}(\Omega) = \int_\Omega \nu_{+,n} dx \leqslant C_2\lambda_n$ and $\nu_{+,n}(\Omega) \to \nu_+(\Omega)$ then $\nu_+(\Omega) \leqslant C_2(\lambda_0 + 1) < \infty$, so that $\sharp\mathcal{S}_{u_+} < \infty$.

Claim 1. If $\mathcal{S}_{u_+} = \emptyset$, then Alternative (i) holds.

Indeed, if $\mathcal{S}_{u_+} = \emptyset$ holds, then in view of Lemma 3.2 with $p = +\infty$ and of the compactness of Ω we have

$$\limsup_{n\to+\infty} \|u^+_{+,n}\|_{L^\infty(\Omega)} < +\infty.$$

Then, by elliptic estimates,

$$\limsup_{n\to\infty} \|u^+_{+,n}\|_{W^{2,r}(\Omega)} < +\infty, \qquad r \in [1, +\infty),$$

and therefore we may extract a subsequence $\{u_{+,n_k}\}$ such that $u_{+,n_k} \to u_+$, for some $u_+ \in \mathcal{E}$. Similarly, if $S_{u_-} = \emptyset$ then there exists a subsequence $u_{-,n_k} \to u_-$, for some $u_- \in \mathcal{E}$, where

$$\mathcal{S}_{u_-} = \{p \in \Omega : \nu_-(\{p\}) \geqslant 4\pi\}.$$

We conclude that if $\mathcal{S}_{u_+} \cup \mathcal{S}_{u_-} = \emptyset$, then $v_n \to v = u_+ - u_-$ in $\mathcal{E}$. Claim 1 is established.

Claim 2. If $\mathcal{S}_{u_+} \cup \mathcal{S}_{u_-} \neq \emptyset$, then Alternative 2 holds. We first assume that $S_{u_+} \neq \emptyset$. In this case, for any $\omega \subset\subset \Omega \setminus \mathcal{S}_{u_+}$ we have

$$\limsup_{n\to+\infty} \|u^+_{+,n}\|_{L^\infty(\omega)} < +\infty,$$

and therefore, there exists $s_+ \in L^\infty_{loc}(\Omega\setminus\mathcal{S}_{u_+})$ such that $\nu_{+,n|_\omega} \to s_+$ in $L^p(\omega)$ for all $p \in [1, +\infty)$. It follows that $\nu_{+|_\omega} = s_+dx$, while the singular part of ν_+ is supported on $\mathcal{S}_{u_+}$. Hence,

$$\nu_+ = s_+ + \sum_{p\in\mathcal{S}_{u_+}} n_{+,p}\delta_p$$

for some $n_{+,p} \geqslant 4\pi$. Similarly

$$\nu_- = s_- + \sum_{p\in\mathcal{S}_{u_-}} n_{-,p}\delta_p$$

where $n_{-,p} \geqslant 4\pi$.

We are left to show that

$$\mathcal{S}_{u_+} = \mathcal{S}_+ \qquad \text{and} \qquad \mathcal{S}_{u_-} = \mathcal{S}_-.$$

Let us start by proving that $\mathcal{S}_+ \subseteq \mathcal{S}_{u_+}$. To this aim assume $p_0 \notin \mathcal{S}_{u_+}$. Then, by Lemma 3.2 there exists a neighborood of p_0 $U \subset \Omega$ such that

$$\limsup_{n\to\infty} \|u^+_{+,n}\|_{L^\infty(U)} < +\infty.$$

Since $v_n = u_{+,n} - u_{-,n} \leqslant u_{+,n} + C$, this implies

$$\limsup_{n\to\infty} \|v_n^+\|_{L^\infty(U)} < +\infty$$

i.e. $p_0 \notin \mathcal{S}_+$. To prove that $\mathcal{S}_{u_+} \subseteq \mathcal{S}_+$ let $p_0 \in \mathcal{S}_{u_+}$. As already seen S_{u_+} coincides with the singular support of ν_+ and consequently the sequence of functions

$$\nu_{+,n} = \lambda_n \int_{I_+} V(\alpha, x, v_n) e^{\alpha v_n} \mathcal{P}(d\alpha) dx$$

is L^∞-unbounded near $p_0 \in \mathcal{S}_{u_+}$. This implies that, for every $r > 0$

$$+\infty = \lim_{n\to\infty} \sup_{B(p_0,r)} \nu_{+,n} \leqslant \lim_n \sup_{B(p_0,r)} C_1 \lambda_n (e^{v_n} + 1).$$

In particular

$$\lim_{n\to\infty} \sup_{B(p_0,r)} v_n = +\infty$$

so that $p_0 \in \mathcal{S}_+$. The proof for $\mathcal{S}_-$ is similar.

In order to prove (2.3) we generalize the approach in [10]. Let $k_n(\alpha, x) = V(\alpha, x, v_n(x))$. In view of (V2), we have $\|k_n\|_{L^\infty(I\times\Omega)} \le C_1$. Therefore, passing to a subsequence, we may assume that k_n converges weak-$*$ in $L^\infty(I \times \Omega)$ to some $k \in L^\infty(I \times \Omega)$. Setting

$$c_n = \frac{\lambda_n}{|\Omega|} \iint_{I\times\Omega} V(\alpha, x, v_n) e^{\alpha v_n},$$

in view of (V3) we may assume that $c_n \to c \in \mathbb{R}$. On the other hand, since v_n is bounded in $W^{1,q}(\Omega)$ for all $q \in [1, 2)$, we may also assume that $v_n \to v \in W^{1,q}(\Omega)$ strongly in $L^r(\Omega)$ for $r \in [1, \infty)$. We fix $\omega \Subset \Omega \setminus \mathcal{S}$ and we take a test function $\varphi \in C^\infty(\omega)$. We have, for all n,

$$\int_\Omega \nabla v_n \cdot \nabla\varphi = \lambda_n \iint_{I\times\Omega} k_n(\alpha, x) e^{\alpha v_n} \varphi\, \mathcal{P}(d\alpha) dx - c_n \int_\Omega \varphi. \tag{3.8}$$

Taking limits, we obtain

$$\int_\Omega \nabla v \cdot \nabla\varphi = \lambda_0 \iint_{I\times\Omega} k(\alpha, x) e^{\alpha v} \varphi\, \mathcal{P}(d\alpha) dx - c_0 \int_\Omega \varphi. \tag{3.9}$$

Since φ is an arbitrary test function supported in ω, we conclude that (2.3) holds true in ω. Since $\omega \Subset \Omega \setminus \mathcal{S}$ is also arbitrary, (2.3) is established on the whole of Ω. □

We proceed towards the proof of Theorem 2.2. Hence we assume that depends on α, v only, $\nabla_x V(\alpha, v) = 0$ and that (V2')–(V3') hold. We denote

$$\tilde\mu_{\pm,n}(dx) = \lambda_n \int_{I_\pm} \frac{V(\alpha, v_n)}{\alpha} e^{\alpha v_n} \mathcal{P}(d\alpha) dx.$$

Since $\tilde\mu_{\pm,n}(\Omega) \leqslant C_2' \lambda_n$, up to subsequences $\tilde\mu_{\pm,n} \overset{*}{\rightharpoonup} \tilde\mu_\pm$ for some Borel measures $\tilde\mu_\pm \in \mathcal{M}(\Omega)$. We first prove a lemma.

Lemma 3.3. *There exists $\tilde s_\pm \in L^\infty_{loc}(\Omega \setminus \mathcal{S}_\pm)$ and $\tilde m_\pm(p) \geqslant 4\pi$, $p \in \mathcal{S}_\pm$, such that*

$$\tilde\mu_\pm = \tilde s_\pm + \sum_{p\in\mathcal{S}_\pm} \tilde m_\pm(p) \delta_p. \tag{3.10}$$

Proof. By definition of $\mathcal{S}_\pm$, for every $\omega \subset\subset \Omega \setminus \mathcal{S}_\pm$ there exists a positive constant $C = C(\omega)$ such that

$$\sup_\omega |v_n| \leqslant C \qquad\qquad \text{for any } n \in \mathbb{N}.$$

It follows that, for any measurable set $E \subset \omega$

$$\tilde{\mu}_{\pm,n}(E) = \lambda_n \iint_{I_\pm \times E} \frac{V(\alpha, v_n)}{\alpha} e^{\alpha v_n} \mathcal{P}(d\alpha) \leqslant C_1' \lambda_n e^C |E|.$$

Hence, the singular parts of $\tilde{\mu}_\pm$ are contained in $\mathcal{S}_\pm$ so that (3.10) holds for some $\tilde{s}_\pm \in L^1(\Omega) \cap L^\infty_{loc}(\Omega \setminus \mathcal{S}_\pm)$ and for some $\tilde{m}_\pm(p) \geqslant 0$, $p \in \mathcal{S}_\pm$. On the other hand, since $\tilde{\mu}_{\pm,n} \geqslant \nu_{\pm,n}$, then $\tilde{m}_\pm(p) = \tilde{\mu}_\pm(\{p\}) \geqslant \nu_+(\{p\}) \geqslant 4\pi$. This completes our proof. □

Let μ_n and μ be as in formulas (2.5) and (2.6). We are in position to prove Part (i) of Theorem 2.2:

Proof of Theorem 2.2. Part (i). To prove that there exists $\zeta_p \in \mathcal{M}(I)$ and $r \in L^1(I \times \Omega)$, $r \geqslant 0$ such that

$$\mu(d\alpha dx) = \sum_{p \in \mathcal{S}} \zeta_p(d\alpha)\delta_p(dx) + r(\alpha, x)\mathcal{P}(d\alpha)dx,$$

it suffices to show that the singular part of μ is supported on $I \times \mathcal{S}$. To this aim, let us take $A \Subset I \times (\Omega \setminus \mathcal{S})$. Then, there exists a constant $C = C(A)$ such that $\|\alpha v_n\|_{L^\infty(A)} \leqslant C$. Hence, for large values of n we obtain

$$\mu_n(A) = \lambda_n \iint_A \frac{V(\alpha, v_n)}{\alpha} e^{\alpha v_n} \leqslant C_1'(\lambda_0 + 1)e^C \iint_A \mathcal{P}(d\alpha)dx$$

so that, on A, μ_n is absolutely continuous. This implies that μ_n does not have singularities on $A \Subset I \times (\Omega \setminus \mathcal{S})$ so that the thesis follows.

Part (ii). We recall from Section 2 that

$$\mu_\alpha = \lambda \frac{V(\alpha, v)}{\alpha} e^{\alpha v}\, dx.$$

We define

$$u_\alpha(x) = G \star \mu_\alpha(x) = \int_\Omega G(x, x')\mu_\alpha(x')dx',$$

where G is the Green's function defined by (3.6). Then,

$$v = \int_I \alpha u_\alpha \mathcal{P}(d\alpha)$$

and $(u_\alpha)_{\alpha \in I}$ satisfies the Liouville type system:

$$-\Delta u_\alpha = \lambda \frac{V(\alpha, v)}{\alpha} \exp\left\{\alpha \int_I \alpha' u_{\alpha'} \mathcal{P}(d\alpha')\right\} - c_\alpha \qquad\qquad \int_\Omega u_\alpha = 0,$$

where

$$c_\alpha = \frac{\lambda}{|\Omega|} \int_\Omega \frac{V(\alpha, v)}{\alpha} \exp\left\{\alpha \int_I \alpha' u_{\alpha'} \mathcal{P}(d\alpha')\right\}.$$

Now we use Suzuki's symmetry argument as introduced in [18, 12, 20]. Let us first observe that μ_α verifies:

$$\nabla \mu_\alpha = \alpha \mu_\alpha \nabla v = \alpha \mu_\alpha \int_I \alpha' \nabla u_{\alpha'} \mathcal{P}(d\alpha') = \alpha \mu_\alpha \int_I \alpha' (\nabla G) \star \mu_{\alpha'} \mathcal{P}(d\alpha'). \tag{3.11}$$

We note that, despite of the general form of the potential V, equation (3.11) is identical to equation (30) in [9]. Equation (3.11) and with Part (i) in Theorem 2.2 are key ingredients necessary to the above mentioned symmetry argument. With such ingredients at hand, the proof of Part (ii) follows exactly as in [9]. For the reader's convenience, we sketch it briefly in what follows.

Let χ be a C^1-vector field over Ω, and define

$$\rho_\chi : \Omega^2 \setminus \{(x,x') \in \Omega^2 : x = x'\} \to \mathbb{R}$$

by

$$\rho_\chi(x,x') = \frac{1}{2}[\chi(x) \cdot \nabla_x G(x,x') + \chi(x') \cdot \nabla_{\chi'} G(x,x')].$$

Recall from Section 2 that

$$\mu_\alpha^n = \lambda_n \frac{V(\alpha, v_n)}{\alpha} e^{\alpha v_n}\, dx.$$

Then, Suzuki's symmetry trick yields the following key identity:

$$\iint_{I\times\Omega} (\text{div } \chi)\mu_\alpha^n \mathcal{P}(d\alpha)dx = -\iint_{I^2} \alpha\alpha' \mathcal{P}(d\alpha)\mathcal{P}(d\alpha') \int\int_{\Omega^2} \rho_\chi(x,x')\mu_\alpha^n \mu_{\alpha'}^n dx dx',$$

For any choice of χ such that ρ_χ is continuous on Ω^2, taking limits in last equality, in view of Part (i) we obtain the identity:

$$\begin{aligned}
&\sum_{p\in\mathcal{S}} \int_I (\text{div } \chi)(p)\zeta_p(d\alpha) + \iint_{I\times\Omega} (\text{div } \chi)(x) r(\alpha,x)\mathcal{P}(d\alpha)dx \\
&\quad = \iint_{I^2} \Big[\sum_{p,q\in\mathcal{S}} \zeta_p(d\alpha)\zeta_q(d\alpha')\rho_\chi(p,q) + \sum_{p\in\mathcal{S}} \zeta_p(d\alpha)\mathcal{P}(d\alpha') \int_\Omega r(\alpha',x')\rho_\chi(p,x')dx' \\
&\quad + \sum_{q\in\mathcal{S}} \zeta_q(d\alpha')\mathcal{P}(d\alpha) \int_\Omega r(\alpha,x)\rho_\chi(x,q)dx + \iint_{\Omega^2} r(\alpha,x) r(\alpha',x')\rho_\chi(x,x')dx dx' \mathcal{P}(d\alpha)\mathcal{P}(d\alpha')\Big].
\end{aligned} \tag{3.12}$$

We fix $p_0 \in \mathcal{S}$ and take an isothermal coordinate chart (ψ, U) satisfying $\psi(p_0) = 0$, $g(X) = e^\xi(dX_1^2 + dX_2^2)$, and $\xi(0) = 0$. Let $B(p_0, 2r) \subset U$ and $B(p_0, 2r) \cup \mathcal{S} = \{p_0\}$. We recall the following expansions of the Green's function:

$$G(X,X') = -\frac{1}{2\pi} \log|X - X'| + \omega(X,X'),$$

$$\nabla_X G(X,X') = -\frac{1}{2\pi} \frac{X - X'}{|X - X'|^2} + \nabla_X \omega(X,X'),$$

$$\nabla_{X'} G(X,X') = \frac{1}{2\pi} \frac{X - X'}{|X - X'|^2} + \nabla_{X'} \omega(X,X'),$$

with ω satisfying

$$\|\omega\|_{L^\infty(B(p_0,2r)^2)} + \|\nabla_X \omega\|_{L^\infty(B(p_0,2r)^2)} + \|\nabla_{X'} \omega\|_{L^\infty(B(p_0,2r)^2)} = O(1)$$

as $r \to 0$. Let $\varphi \in C(\Omega)$ be a cut-off function such that $\varphi \equiv 1$ in $B(p_0, r)$ and $\varphi \equiv 0$ in $\Omega \setminus B(p_0, 2r)$. We choose $\chi(X) = 2X\varphi$. With this choice of χ we may write:

$$\rho_\chi(X,X') = (-\frac{1}{2\pi} + \eta)\varphi$$

where $\eta(X,X')$ is a continuous function on Ω^2. Moreover, we have

$$\operatorname{div}\,\chi(X) = |g|^{-1/2}\partial X_j(|g|^{1/2}(\chi)^j) = 4 + O(X).$$

Consequently, taking limits for each term in (3.12) as $r \downarrow 0$ we derive:

$$\begin{aligned}
&\sum_{p\in\mathcal{S}}\int_I (\operatorname{div}\,\chi)(p)\zeta_{p_0}(d\alpha) \to 4\int_I \zeta_{p_0}(d\alpha);\\
&\left|\iint_{I\times\Omega}(\operatorname{div}\,\chi)(x)r(\alpha,x)\mathcal{P}(d\alpha)dx\right| = o(1);\\
&\iint_{I^2}\sum_{p,q\in\mathcal{S}}\rho_\chi(p,q)\zeta_p(d\alpha)\zeta_q(d\alpha') \to -\frac{1}{2\pi}\iint_{I^2}\zeta_{p_0}(d\alpha)\zeta_{p_0}(d\alpha');\\
&\iint_{I^2}\sum_{p\in\mathcal{S}}\zeta_p(d\alpha)\int_\Omega r(\alpha',x')\rho_\chi(p,x')dx'\mathcal{P}(d\alpha') = o(1).
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
&\iint_{I^2}\sum_{q\in\mathcal{S}}\zeta_q(d\alpha')\int_\Omega r(\alpha,x)\rho_\chi(x,q)\mathcal{P}(d\alpha) = o(1)\\
&\iint_{I^2}\iint_{\Omega^2} r(\alpha,x)r(\alpha',x')\rho_\chi(x,x')dxdx'\mathcal{P}(d\alpha)\mathcal{P}(d\alpha') = o(1).
\end{aligned}$$

Inserting into (3.12) we conclude the proof of Part (ii).

Part (iii). We provide the proof for the "I_+-case", the proof for I_- being exactly the same. Let $\varepsilon > 0$ and let $\varphi \in C(\Omega)$, $\psi \in C(I)$, $0 \leqslant \psi(\alpha) \leqslant 1$, $\psi \equiv 1$ on I_+, $\psi \equiv 0$ on $[-1,-\varepsilon]$. We have

$$\begin{aligned}
\iint_{I\times\Omega}|\alpha|\varphi(x)\psi(\alpha)\mu_n(d\alpha dx) = &\int_\Omega \varphi(x)\nu_{+,n}(dx)\\
&+ \lambda_n\iint_{[-\varepsilon,0]\times\Omega} V(\alpha,v_n)\psi(\alpha)\varphi(x)\,dx\mathcal{P}(d\alpha).
\end{aligned} \tag{3.13}$$

Taking limits on the left-hand side of (3.13) as $n\to\infty$, by (2.7) we have

$$\begin{aligned}
&\iint_{I\times\Omega}|\alpha|\varphi(x)\psi(\alpha)\mu_n(d\alpha dx)\\
&\to \sum_{p\in\mathcal{S}}\int_I|\alpha|\psi(\alpha)\varphi(p)\zeta_p(d\alpha) + \iint_{I\times\Omega}|\alpha|\varphi(x)\psi(\alpha)r(\alpha,x)\mathcal{P}(d\alpha)dx.
\end{aligned}$$

Moreover,

$$\int_I|\alpha|\psi(\alpha)\zeta_p(d\alpha) = \int_{I_+}|\alpha|\zeta_p(d\alpha) + \int_{[-\varepsilon,0]}|\alpha|\psi(\alpha)\zeta_p(d\alpha),$$

and

$$\begin{aligned}
&\iint_{I\times\Omega}|\alpha|\varphi(x)\psi(\alpha)r(\alpha,x)\mathcal{P}(d\alpha)dx\\
&= \iint_{I_+\times\Omega}|\alpha|\varphi(x)r(\alpha,x)\mathcal{P}(d\alpha)dx + \iint_{[-\varepsilon,0]\times\Omega}|\alpha|\varphi(x)\psi(\alpha)r(\alpha,x)\mathcal{P}(d\alpha)dx.
\end{aligned}$$

We note that

$$0 \leqslant \int_{[-\varepsilon,0]}|\alpha|\psi(\alpha)\zeta_p(d\alpha) \leqslant \varepsilon\int_{[-\varepsilon,0]}\psi(\alpha)\zeta_p \le c_1\varepsilon.$$

Furthermore,

$$\left| \iint_{[-\varepsilon,0]\times\Omega} |\alpha|\varphi(x)\psi(\alpha)r(\alpha,x)\mathcal{P}(d\alpha)dx \right| \leqslant \varepsilon\|\varphi\|_\infty \iint_{I\times\Omega} r(\alpha,x)\mathcal{P}(d\alpha)dx.$$

Analogously, by passing to the limit as $n\to\infty$ on the right hand side of (3.13) we have

$$\int_\Omega \varphi(x)\nu_{+,n}(dx) \to \sum_{p\in\mathcal{S}_+} n_{+,p}\varphi(p) + \int_\Omega s_+\varphi.$$

Moreover, in view of (V2'),

$$\lambda_n \int_{[-\varepsilon,0]} |\alpha|\psi(\alpha) \int_\Omega \frac{|V(\alpha,v_n)|}{|\alpha|}\varphi(x)e^{\alpha v_n}\, dx\mathcal{P}(d\alpha) \leqslant C_2'\lambda_n\varepsilon\|\varphi\|_\infty.$$

Hence, combining the estimates above we obtain

$$\begin{aligned}
&\sum_{p\in\mathcal{S}_+} n_{+,p}\varphi(p) + \int_\Omega s_+\varphi + c_1\varepsilon\|\varphi\|_\infty \\
&= \sum_{p\in\mathcal{S}} \int_{I_+} |\alpha|\zeta_p(d\alpha)\varphi(p) + \iint_{I_+\times\Omega} |\alpha|\varphi(x)r(\alpha,x)\mathcal{P}(d\alpha)dx + c_2\varepsilon\|\varphi\|_\infty,
\end{aligned}$$

where c_1 and c_2 are constants uniformly bounded with respect to ε. So, by passing to the limit as $\varepsilon\to 0^+$ in last equality, we obtain

$$\sum_{p\in\mathcal{S}_+} n_{+,p}\varphi(p) + \int_\Omega s_+\varphi = \sum_{p\in\mathcal{S}} \int_{I_+} |\alpha|\zeta_p(d\alpha)\varphi(p) + \iint_{I_+\times\Omega} |\alpha|\varphi(x)r(\alpha,x)\mathcal{P}(d\alpha)dx. \tag{3.14}$$

Now, assume $\varphi\in C(\Omega)$ with $\mathrm{supp}\varphi\subset\Omega\setminus\mathcal{S}$. Then,

$$\int_\Omega s_+\varphi = \int_\Omega \varphi(x)\int_{I_+} |\alpha|r(\alpha,x)\mathcal{P}(d\alpha)dx$$

so that for almost every $x\in\Omega$

$$s_+ = \int_{I_+} |\alpha|r(\alpha,x)\mathcal{P}(d\alpha)$$

since $\mathcal{S}$ is null set with respect to dx. By (3.14) this implies

$$\sum_{p\in\mathcal{S}_+} n_{+,p}\varphi(p) = \sum_{p\in\mathcal{S}} \int_{I_+} |\alpha|\zeta_p(d\alpha)\varphi(p). \tag{3.15}$$

Now let us fix $p_0\in\mathcal{S}_+$ and let $\varphi\in C(\Omega)$ be such that $\mathrm{supp}\varphi\subset B_\rho(p_0)$, with $B_\rho(p_0)\cap\mathcal{S}=\{p_0\}$ and verifying $\varphi(p_0)=1$. By (3.15) then we have

$$n_{+,p_0} = \int_{I_+} |\alpha|\zeta_{p_0}(d\alpha)$$

for any $p_0\in\mathcal{S}_+$. To conclude, for $p_0\in\mathcal{S}_-\setminus\mathcal{S}_+$, let us assume $\varphi\in C(\Omega)$ as above. By (3.15) we get

$$\int_{I_+} |\alpha|\zeta_{p_0}(d\alpha) = 0.$$

This completes our proof. $\square$

4 The cases of physical interest

In this section we consider the special cases of (1.1) which are of interest in statistical turbulence. Namely, we consider the special case $V(\alpha, x, v) = V_1(\alpha, v)$, where V_1 is given by (1), corresponding to Sawada and Suzuki's equation (1.2), and the special case $V(\alpha, x, v) = V_2(\alpha, v)$, where V_2 is given by (1.5), corresponding to Neri's equation (1.6).

It is clear that in both cases V_1, V_2 satisfy (V0)–(V1). We claim that (V2')–(V3') are also satisfied. Indeed, by Jensen's inequality we have

$$\int_\Omega e^{\alpha v} \geq |\Omega|$$

for all $v \in \mathcal{E}$ and for all $\alpha \in I$. Therefore, we have

$$\begin{aligned} 0 \leq \alpha^{-1} V_1(\alpha, v) &= \frac{1}{\int_\Omega e^{\alpha v}} \leq \frac{1}{|\Omega|} \\ 0 \leq \alpha^{-1} V_2(\alpha, v) &= \frac{1}{\iint_{I\times\Omega} e^{\alpha v}} \leq \frac{1}{|\Omega|}, \end{aligned}$$

where we used $\mathcal{P}(I) = 1$ in the last inequality. Hence (V2') is satisfied with $C_1' = |\Omega|^{-1}$ in both cases. On the other hand, we have

$$\begin{aligned} &\iint_{I\times\Omega} |V_1(\alpha, v)| e^{\alpha v}\, dx \mathcal{P}(d\alpha) = \iint_{I\times\Omega} \frac{|\alpha| e^{\alpha v}}{\int_\Omega e^{\alpha v}}\, dx \mathcal{P}(d\alpha) = \int_I |\alpha| \mathcal{P}(d\alpha) \leq 1 \\ &\iint_{I\times\Omega} |V_2(\alpha, v)| e^{\alpha v}\, dx \mathcal{P}(d\alpha) = \iint_{I\times\Omega} \frac{|\alpha| e^{\alpha v}}{\iint_{I\times\Omega} e^{\alpha v}}\, dx \mathcal{P}(d\alpha) \leq 1. = \int_I |\alpha| \mathcal{P}(d\alpha) \leq 1 \end{aligned}$$

Hence, (V3') is also satisfied in both special cases with $C_2' = 1$. We conclude that Theorem 2.1 and Theorem 2.2 hold true for solution sequences to (1.2) and (1.6). In other words, (1.2) and (1.6) are similar from the point of view of blow-up.

If $k \equiv 0$ in (2.3) we say that residual vanishing occurs. In the following theorem we provide a sufficient condition for residual vanishing, in the special case where $V = V_2$ has the form (1.5). The proof is an adaptation of an argument from [13] to our case.

Theorem 4.1. *If* supp$\mathcal{P} \cap \{-1, 1\} \neq 0$ *and if there exists* $p \in \mathcal{S}_+ \setminus \mathcal{S}_-$ *such that* $n_{+,p} > 4\pi$ *then* $k \equiv 0$ *in* (2.3).

Proof. We consider the case where $\mathcal{P}([1-\delta, 1]) > 0$ for all $0 < \delta \ll 1$ and $p \in \mathcal{S}_+ \setminus \mathcal{S}_-$. The remaining cases are is analogous. For every fixed $T > 0$ we truncate the Green's function

$$G^T(x, \cdot) = \min\{T, G(x, \cdot)\} \in C(\Omega).$$

Then,

$$\begin{aligned} u_{+,n}(x) = \int_\Omega G(x, \cdot)\nu_{+,n} \geqslant \int_\Omega G^T(x, \cdot)\nu_{+,n} \to \int_\Omega G^T(x, \cdot)\nu_+ \\ = n_{+,p} G^T(x, p) + \int_\Omega G^T(x, \cdot)(\nu_+ - n_{+,p}\delta_p) \\ \geqslant n_{+,p} G^T(x, p) - C. \end{aligned}$$

Hence,

$$\liminf_n u_{+,n}(x) \geqslant n_{+,p} G^T(x, p) - C.$$

Letting $T \to \infty$,

$$\liminf_n u_{+,n} \geqslant n_{+,p} G(x,p) - C.$$

On the other hand it is well known that in a local chart on $B_\rho = B_\rho(p)$

$$G(x,p) \geqslant \frac{1}{2\pi} \log \frac{1}{|x|} - C.$$

Therefore,

$$\exp\{\alpha u_{+,n}(x)\} \geqslant \exp\{\alpha n_{+,p}(\frac{1}{2\pi} \log \frac{1}{|x|} - C)\} \simeq \left(\frac{1}{|x|}\right)^{\alpha n_{+,p}/2\pi}.$$

Since $p \notin \mathcal{S}_-$, then $u_-(x) \leqslant C$ in $B_\rho(p)$ whenever ρ is suitable small. We observe that the function $\int_\Omega e^{\alpha v} dx$ is increasing with respect to $\alpha > 0$. In fact, differentiation with respect to α yields:

$$\begin{aligned} \frac{d}{d\alpha} \int_\Omega e^{\alpha v} dx &= \int_\Omega v e^{\alpha v} dx \geqslant \int_{v \geqslant 0} v dx - \int_{v<0} e^{\alpha v}(-v) dx \\ &> \int_{v \geqslant 0} v dx - \int_{v<0} (-v) dx = \int_\Omega v = 0. \end{aligned} \tag{4.1}$$

Using this fact, we conclude that,

$$\begin{aligned} \liminf_n \iint_{I \times \Omega} e^{\alpha v_n} &= \liminf_n \iint_{I \times \Omega} e^{\alpha(u_{+,n} - u_{-,n})} \\ &\geqslant \mathcal{P}([1-\delta, 1]) \liminf_n \int_\Omega e^{(1-\delta)(u_{+,n} - u_{-,n})} \\ &\geqslant \mathcal{P}([1-\delta, 1]) e^{-C} \liminf_n \int_{B_\rho(p)} e^{(1-\delta) u_{+,n}} \\ &\geqslant e^C \mathcal{P}([1-\delta, 1]) \int_{B_\rho(p)} \left(\frac{1}{|x|}\right)^{(1-\delta) n_{+,p}/2\pi}. \end{aligned}$$

Choosing δ such that $(1-\delta) n_{+,p} > 4\pi$ we conclude the proof. □

We now consider the Trudinger-Moser type inequalities associated to (1.2) and (1.6). We recall that (1.2) is the Euler-Lagrange equation for the functional

$$\mathcal{J}_\lambda(v) = \frac{1}{2} \|\nabla v\|_2^2 - \lambda \int_I \log \left(\int_\Omega e^{\alpha v} \right) \mathcal{P}(d\alpha), \tag{4.2}$$

defined for $v \in \mathcal{E}$. For $\mathcal{P} = \delta_1$, the functional $\mathcal{J}_\lambda(v)$ is the functional

$$\frac{1}{2} \|\nabla v\|_2^2 - \lambda \log \left(\int_\Omega e^v \right), \tag{4.3}$$

whose Euler-Lagrange equation is the standard meanfield equation (1.3). In view of the classical Trudinger-Moser inequality, as established in [4]:

$$\sup \left\{ \int_\Omega e^{4\pi v^2} \,:\, v \in \mathcal{E},\ \|\nabla v\|_2 \leqslant 1 \right\} < +\infty, \tag{4.4}$$

where the constant 4π is sharp, the functional (4.3) is bounded from below on $\mathcal{E}$ if and only if $\lambda \leq 8\pi$. In [13], as an application of the blow-up analysis, it is shown that if $\mathcal{P} = t\delta_1 + (1-t)\delta_{-1}$,

$t \in [0,1]$, then the optimal value of λ which ensures boundedness from below of the functional (4.2) is improved to $8\pi \min\{t^{-1}, (1-t)^{-1}\}$. By using the blow-up analysis developed in [9] and similar arguments one may check that (4.2) is bounded from below if

$$\lambda \leq \frac{8\pi}{\max\{\int_{I_+} \alpha^2 \mathcal{P}(d\alpha), \int_{I_-} \alpha^2 \mathcal{P}(d\alpha)\}}.$$

This value is however is in general not optimal. The best constant is actually given by

$$\inf \left\{ \frac{8\pi \mathcal{P}(K_\pm)}{\left(\int_{K_\pm} \alpha \mathcal{P}(d\alpha)\right)^2} : \ K_\pm \subset I_\pm \cap \operatorname{supp} \mathcal{P} \right\},$$

see [16].

In view of such "improved" Trudinger-Moser inequalities, it is natural to seek analogous results for (1.6). More precisely, we note that (1.6) is the Euler-Lagrange equation for the functional

$$\mathcal{K}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \left(\iint_{I \times \Omega} e^{\alpha v} \, dx \mathcal{P}(d\alpha) \right). \tag{4.5}$$

However, it is not difficult to check that such an improvement does *not* hold for $\mathcal{K}_\lambda$, that is, $\mathcal{K}_\lambda$ is bounded from below if and only if $\lambda \leq 8\pi$. The "if" part was already observed in [8]. Indeed, from

$$\alpha v \leqslant \frac{\|\nabla v\|_2^2}{16\pi} + 4\pi\alpha^2 \frac{v^2}{\|\nabla v\|_2^2}$$

we derive using (4.4) that

$$\log \left(\int_\Omega \int_{[-1,1]} e^{\alpha v} \mathcal{P}(d\alpha) dx \right) \leqslant \frac{1}{16\pi} \|\nabla v\|_2^2 + K,$$

where K is independent of $v \in \mathcal{E}$. See also [21]. Therefore $\mathcal{K}_\lambda$ is bounded below if $\lambda \leq 8\pi$. On the other hand, differently from what happens for the functional (4.2), the value 8π is also optimal, provided that $\operatorname{supp} \mathcal{P} \cap \{-1, 1\} \neq \emptyset$. Indeed, the following holds:

Theorem 4.2. *Let* $\operatorname{supp} \mathcal{P} \cap \{-1,1\} \neq \emptyset$. *Then, the functional* $\mathcal{K}_\lambda(v)$ *is bounded from below on* $\mathcal{E}$ *if and only if* $\lambda \leqslant 8\pi$.

Proof. We need only prove that

$$\inf_{v \in \mathcal{E}} \mathcal{K}_\lambda(v) = -\infty, \qquad \forall \lambda > 8\pi. \tag{4.6}$$

Using (4.1), for any $\delta > 0$ we have:

$$
\begin{aligned}
\mathcal{K}_\lambda(v) \quad &= \frac{1}{2}\|v\|^2 - \lambda \log\left(\iint_{I\times\Omega} e^{\alpha v}\mathcal{P}(d\alpha)dx\right)\\
&= \frac{1}{2}\|v\|^2 - \lambda \log\left(\int_{1-\delta}^{1}\int_\Omega e^{\alpha v}\mathcal{P}(d\alpha)dx + \int_{-1}^{1-\delta}\int_\Omega e^{\alpha v}\mathcal{P}(d\alpha)dx\right)\\
&= \frac{1}{2}\|v\|^2 - \lambda \log\left(\int_{1-\delta}^{1}\int_\Omega e^{\alpha v}\mathcal{P}(d\alpha)dx\right)\\
&-\lambda\log\left(1+\frac{\int_{-1}^{1-\delta}\int_\Omega e^{\alpha v}\mathcal{P}(d\alpha)dx}{\int_{1-\delta}^{1}\int_\Omega e^{\alpha v}\mathcal{P}(d\alpha)dx}\right)\\
&\leqslant \frac{1}{2}\|v\|^2 - \lambda \log\left(\int_\Omega e^{(1-\delta)v}dx\right) - \lambda\log(\mathcal{P}([1-\delta,1]))\\
&-\lambda\log\left(1+\frac{\int_{-1}^{1-\delta}\int_\Omega e^{\alpha v}\mathcal{P}(d\alpha)dx}{\int_{1-\delta}^{1}\int_\Omega e^{\alpha v}\mathcal{P}(d\alpha)dx}\right)\\
&\leqslant \frac{1}{(1-\delta)^2}\left[\frac{1}{2}\|(1-\delta)v\|^2 - \lambda(1-\delta)^2\log\left(\int_\Omega e^{(1-\delta)v}dx\right)\right] - \lambda\log(\mathcal{P}([1-\delta,1])).
\end{aligned}
$$

Hence, for $\lambda(1-\delta)^2 > 8\pi$, the right hand side of last inequality is unbounded from below (see [4]) and so

$$\inf_{v\in\mathcal{E}} \mathcal{K}_\lambda(v) = -\infty \qquad \text{for any } \lambda > \frac{8\pi}{(1-\delta)^2}.$$

Since $\delta > 0$ is arbitrary, (4.6) follows. ◻

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